WEAK TURBULENCE IN MEDIA WITH A DECAY SPECTRUM

V. E. Zakharov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 35–39, 1965

The theory of weak turbulence of a plasma has been investigated in many papers [1–5]. It has been established that weak turbulence may be described by means of the kinetic wave equations. Here the collision term in the kinetic equation is the sum of two substantially different components. The first of these has the character of nonlinear wave damping and differs from zero in those cases where interaction between waves and particles is significant. It has a comparatively simple mathematical nature and can be analyzed. The second component is specifically a collision term, it depends closely on the form of the spectrum in the medium and describes the exchange of energy between different groups of waves. The case when the second component plays the principal role in the collision term has scarcely been studied. The present paper is devoted to a study of this case.

The analysis is carried out for a simple isotropic model of a medium with an almost linear dispersion law, but with a positive second derivative; we shall call such a spectrum a decay spectrum. This model is much closer to reality than the model considered in [6]. The results obtained from this model are evidently fairly general in character and express substantially the regularity of behavior of weak turbulence in media with a weak decay spectrum. The basic result of the paper is as follows: apart from the Rayleigh-Jeans solution, there exists another solution which reduces the collision term to zero. This solution corresponds to a process which is substantially nonequilibrium, and may be realized in actual problems, where there are always wave sources or transfer terms playing the same part, only in cases where there is wave damping in the medium with a coefficient which increases fairly rapidly into the region of large k. Here the universal character, as it were, of the nonequilibrium process is realized.

NOTATION

- k wave vector;
- ε a parameter characteristic of the dispersion;
- ω_k wave frequency;
- γ_k density of wave sources;
- $V_{kk'k''}$ the matrix element describing the wave interactions;
- $\Gamma(s)$ gamma function;
- u a variable describing the medium;
- k_0 boundary of instability region;
- a_k complex wave amplitude;
- γk^α damping decrement;
- n_k wave density in k space;
- k_1 boundary of the region of transparency;
- N_k wave density in spherical coordinates;
- ν instability increment.

We consider waves in a medium described by a scalar function of coordinates and time u. This quantity obeys the equation

$$\frac{\partial^2 u}{\partial t^2} - (\Delta - \varepsilon\Delta\Delta)\, u = \Delta u^2 \qquad (1)$$

$(\varepsilon > 0)$ (Δ — Laplacian operator).

We carry out a Fourier transformation with respect to coordinates

$$\frac{\partial^2 u_{\mathbf{k}}}{\partial t^2} + \omega_{\mathbf{k}}^2 u_{\mathbf{k}} = k^2 \int u_{\mathbf{k}'} u_{\mathbf{k}-\mathbf{k}'} d\mathbf{k}' \qquad (\omega_{\mathbf{k}}^2 = k^2 + \varepsilon k^4). \quad (2)$$

We pass to the new variables, complex wave amplitudes,

$$a_{\mathbf{k}} = \frac{u_{\mathbf{k}} + i u_{\mathbf{k}}^{\cdot}}{\sqrt{2k\omega_{\mathbf{k}}}}.$$

We obtain the equation for these quantities

$$a_{\mathbf{k}}^{\cdot} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = -i \int V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} (a_{\mathbf{k}'} a_{\mathbf{k}''} \delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} + 2a_{\mathbf{k}'}^* a_{\mathbf{k}''} \delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''} + a_{\mathbf{k}'}^* a_{\mathbf{k}''}^* \delta_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''})\, d\mathbf{k}' d\mathbf{k}'', \qquad (3)$$

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} = \frac{1}{\sqrt{8}} \left(\frac{k^2 k'^2 k''^2}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'} \omega_{\mathbf{k}''}} \right)^{1/2}. \qquad (4)$$

We shall consider the case $k \ll 1/\sqrt{\varepsilon}$. Here we may set

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx \frac{1}{\sqrt{8}} (|\mathbf{k}|\,|\mathbf{k}'|\,|\mathbf{k}''|)^{1/2}. \qquad (5)$$

In order to obtain the kinetic equation, we must apply the theory of perturbations to Eq. (3). We shall determine its criterion of applicability. To do this, we carry out the change of variable

$$a_{\mathbf{k}} = c_{\mathbf{k}} \exp(-i\omega_{\mathbf{k}} t)$$

$$\frac{\partial c_{\mathbf{k}}}{\partial t} = -i \int V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} [c_{\mathbf{k}'} c_{\mathbf{k}''} \exp(it(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}))\, \delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} + \ldots]\, d\mathbf{k}' d\mathbf{k}''.$$

We choose $c_{\mathbf{k}}^\circ = c\delta_{\mathbf{k}-\mathbf{k}_0}$. Then

$$c_{\mathbf{k}}^{(1)} = c_1 \delta_{\mathbf{k}-2\mathbf{k}_0}, \qquad c_1 = \frac{V_{2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0}}{\omega_{2\mathbf{k}_0} - \omega_{\mathbf{k}_0}} = \frac{c^2}{4\varepsilon |\mathbf{k}_0|^{3/2}}.$$

The condition of applicability of perturbation theory is

$$c_1 \ll c, \quad \text{or} \quad n_{\mathbf{k}} \ll 16\varepsilon^2 k^3 \ \text{for} \ n_{\mathbf{k}} = |c_{\mathbf{k}}|^2. \qquad (6)$$

Condition (6) shows that the less the wave spectrum departs from the linear, the smaller is the allowable wave amplitude for which perturbation theory and the kinetic equation may be used. This is connected with the fact that for spectra which are close to being linear, resonance interactions between waves which do not lead to a randomizing of phases may play a large part.

The kinetic equation for the problem under consideration has the form

$$\frac{\partial n_{\mathbf{k}}}{\partial t} + 2\gamma_{\mathbf{k}} n_{\mathbf{k}} = \frac{\pi}{2} \int (|\mathbf{k}|\,|\mathbf{k}'|\,|\mathbf{k}''|)^{1/2} \{\delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} \delta_{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''}} (n_{\mathbf{k}'} n_{\mathbf{k}''} - 2n_{\mathbf{k}} n_{\mathbf{k}'}) + 2\delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''} \delta_{\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''}} (n_{\mathbf{k}'} n_{\mathbf{k}''} + n_{\mathbf{k}} n_{\mathbf{k}''} - n_{\mathbf{k}} n_{\mathbf{k}'})\}\, d\mathbf{k}' d\mathbf{k}''. \quad (7)$$

We shall seek spherically symmetrical solutions of this equation. We introduce the new quantity $N_k = k^2 n_k$. After averaging over angles, we have

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + 2\gamma_{\mathbf{k}} N_{\mathbf{k}} = 2\pi^4 \operatorname{st}(N, N) = 2\pi^4 \Big\{ \int_0^k N_{k'} N_{k-k'} dk' - 4N_k \int_0^k N_{k'} dk' + 2\int_0^\infty N_{k'} N_{k+k'} dk' - \frac{4N_k}{k^2} \int_0^k k^2 N_k dk - \frac{8N_k}{k} \int_k^\infty k N_k dk \Big\}.$$

In averaging we assume that $\omega_k = |k|$ approximately. In these equations $2\gamma_k N_k$ is the density of wave sources formally introduced.

We now investigate the properties of the operator st(N, N). It is clear that it is defined for functions decreasing more rapidly than $1/k^2$ for $k \to \infty$. At first sight, it seems that these functions must diverge for small k. However, this is not so. Actually, all the terms which diverge for small k are grouped together in the integral

$$2\int_0^{\frac{1}{2}k} N_{k'}\left(N_{k-k'} - 2N_k + N_{k+k'}\right) dk' \qquad \left(k'^2 N_{k'} \frac{\partial^2 N_k}{\partial k^2}\right).$$

For small k the integrand expression has the form indicated in brackets, i.e., divergences are confined to two orders.

Thus, the region of definition for the operator st(N, N) will be functions which decrease more rapidly than $1/k^2$ for $k \to \infty$ and increase more slowly than $1/k^3$ for $k \to 0$. Formally, the solution

$$N_k = T_k \qquad (T = \text{const}) \tag{8}$$

satisfies the equation st(N, N) = 0.

This solution is the Rayleigh-Jeans distribution. However, such solutions do not enter into the defining region of the operator and one cannot linearize on a background of these solutions. Physically, this means that in considering these solutions quantum corrections must be taken into account.

However, the equation st(N, N) = 0 has other solutions appropriate to the defining region of the operator. We shall seek a solution in the form

$$N_k = \frac{L}{k^s} \qquad (2 < s < 3).$$

Then

$$\mathrm{st}(N, N) = \frac{L^2}{k^4} F(s) \qquad (L = \text{const}).$$

Here

$$F(s) = \frac{\Gamma^2(1-s)}{\Gamma(2-2s)} - \frac{4}{1-s} + 2\frac{\Gamma(1-s)\Gamma(2s-1)}{\Gamma(s)} - \frac{8}{s-2} + \frac{4}{3-s}. \tag{9}$$

Investigation shows that $F(3) = +\infty$, $F(2) = -\infty$. This means that the function F(s) must have a zero in the interval $2 < s < 3$.

Calculation shows that the zero of F(s) is s = 2.5. Thus the equation st(N, N) = 0 has the solution

$$N_k = Ak^{-2.5}. \tag{10}$$

We shall now consider the problem with wave sources and determine under what conditions a solution close to solution (10) may be realized:

$$2\gamma_k N_k = 2\pi^4\, \mathrm{st}(N, N). \tag{11}$$

Usually the estimate

$$N \sim \gamma / \pi^4 k_0 \tag{12}$$

is applied to this equation.

Here γ is the characteristic value of the increment, and k_0 is the characteristic dimension in k space in which the function γ varies.

However, this estimate applies only in the case when the instability and damping regions are not divided by a region of transparency. In reality the opposite is most frequently the case. We shall show that in just this case a solution of type (10) may be employed.

For a start, we note that if equation (10) is multiplied by k and integrated from zero to infinity, then the right hand side of equation (11) becomes zero. We obtain

$$\int_0^\infty k\gamma_k N_k dk = 0. \tag{13}$$

This relation expresses the law of conservation of energy.

Let there now exist an instability region with a characteristic increment ν and characteristic dimension k_0, and, in addition, let damping of the form

$$\gamma_k = \gamma\pi^{-4}k^{\alpha} \qquad (\alpha > 1/2)$$

exist.

The meaning of this condition will be clear from what follows. We shall designate the solution in the region of small values of k M_k; N_k is the solution in the remaining region. Extending N_k formally into the region of small k, we may write equation (11) in the form

$$\gamma k^{\alpha} N_k = \int_0^k N_{k'}N_{k-k'}dk' - 4N_k\int_0^k N_{k'}dk' + 2\int_0^\infty N_{k'}N_{k+k'}dk' - \frac{4N_k}{k^2}\int_0^k k'^2 N_{k'}dk' - \frac{8N_k}{k}\int_k^\infty k' N_{k'}dk' + 2\int_0^{k_0}(M_{k'} + N_{k'})(N_{k-k'} - 2N_k + N_{k+k'})\,dk' - \frac{4N_k}{k^2}\int_0^{k_0} k'^2 (M_{k'} - N_{k'})\,dk'.$$

For $k \gg k_0$ the last two terms may be transformed to

$$A\left(\frac{\partial^2 N_k}{\partial k^2} - \frac{4N_k}{k^2}\right) \qquad \left(A = \int_0^{k_0} k'^2 (M_{k'} - N_{k'})\,dk'\right). \tag{14}$$

If γ is small enough, we may seek a solution in the form

$$N_k = Bk^{-2.5}. \tag{15}$$

Then the first terms of the equation are of order B^2k^{-4}; terms of type (14) are of order $ABk^{-4.5}$, and consequently their effect may be neglected for large k. A solution of type (15) will be valid right up to those values of k at which none of the integrals entering into the collision term is comparable with the damping term, i.e., to k_1 defined by the relation

$$\gamma k_1^{\alpha-2.5}B \sim B^2 k_1^{-4}, \quad \text{or} \quad k_1 \sim \left(\frac{B}{\gamma}\right)^{1/\alpha+1.5}. \tag{16}$$

The solution decreases rapidly for $k > k_1$.

We use relation (13) to determine the quantity B. We have

$$\nu k_0^2 \frac{B}{k_0^{2.5}} \sim \gamma B \int_0^{k_1} k^{-1.5+\alpha} dk. \quad (17)$$

All this is correct if the chief contribution to the integral on the right hand side of relation (17) comes from large k (otherwise neglecting the effect of the instability region is not legitimate). We thus obtain $\alpha > 1/2$. It follows from (17) that

$$B \sim \gamma^{-\frac{1}{\alpha-1/2}} \left[\frac{\nu(\alpha - 1/2)}{k_0^{1/2}}\right]^{\frac{\alpha+3/2}{\alpha-1/2}}. \quad (18)$$

All this is true if $k_1 \gg k_0$. Calculation of k_1 gives the condition

$$\nu(\alpha - 1/2) \gg \gamma k_0^{\alpha}. \quad (19)$$

This inequality may be satisfied for small enough γ.

Comparison of (18) and (12) shows that estimates of solutions in these two limiting cases are significantly different.

It is clear from the above that a solution of type (10) may be realized in the presence of two conditions: a wide region of transparency and a fairly rapidly increasing damping coefficient. The latter requirement demands an essentially nonequilibrium problem. Under these conditions the solution in the transparent region has a universal form. A similar phenomenon is observed in ordinary turbulence, where a Kolmogorov spectrum is established in the region of transparency. However, the mechanisms determining the universal solution in these two cases are significantly different. To be specific, in the case of ordinary fluid turbulence, scales of the same order interact with each other, so that we may introduce the concept of energy flow through the turbulence spectrum and obtain the spectrum from dimensional considerations. In our case all scales interact simultaneously. Here the solution can not be obtained from dimensional considerations; generally speaking, it depends on the character of the wave interaction.

We shall consider the possibility of generalizing the results obtained above. In the general case decay turbulence is characterized by two factors: the wave spectrum ω_k and the matrix element $V_{kk'k''}$ describing the interaction; in addition ω_k is a positive function which is convex downwards and $V_{kk'k''}$ is a positive function which is symmetric with respect to the last two indices. We choose

$$\omega_{\mathbf{k}} = |\mathbf{k}|^s \, (s > 1), \qquad V_{\mathbf{kk'k''}} = (|\mathbf{k}|\,|\mathbf{k'}|\,|\mathbf{k''}|)^t. \quad (20)$$

Then, considering isotropic solutions of the kinetic equation, we may pass to the variable $\omega = k^s$. We obtain

$$\frac{\partial N_\omega}{\partial t} + 2\gamma_\omega N_\omega = 2\pi^4 s^2 \omega^{t-1/s} \Big\{ \int_0^{\omega} \omega'^p (\omega - \omega')^p N_{\omega'} N_{\omega-\omega'} d\omega' -$$
$$- 2N_\omega \int_0^{\omega} \omega'^p [(\omega - \omega')^p + (\omega + \omega')^p] N_{\omega'} d\omega' + \quad (21)$$
$$+ 2\int_0^{\infty} \omega'^p (\omega + \omega')^p N_{\omega'} N_{\omega+\omega'} d\omega' -$$
$$- 2N_\omega \int_0^{\infty} \omega'^p [(\omega + \omega')^p - (\omega - \omega')^p] N_{\omega'} d\omega' \Big\}$$
$$\left(p = \frac{t - s + 2}{s}\right).$$

Just as in the case considered above, the structure of the first three integrals in the collision term is such that the divergences at zero are reduced by two orders. We shall seek a solution which reduces the collision term to zero in the form

$$N_\omega = L / \omega^q.$$

It is not difficult to see that the allowable values of q lie in the interval

$$2p < q < p + 3. \quad (22)$$

Hence it follows that $p < 3$. The result of applying the collision term to the function q is the expression

$$\frac{L^2}{\omega^{2q-2p+1}} Q(q),$$

where the function Q(q) tends to infinity at the boundaries of the interval (22). Elementary investigation shows that for $q = p + 3$ the infinity is always positive, while for $q = 2p$ the sign of the infinity is opposite to the sign of p. The requirement that these infinities should have different signs, sufficient for the existence of a power solution, leads to the condition $0 < p < 3$. Hence we obtain

$$s - 2 < t < 4s - 2.$$

For such conditions on t and s the equations of decay turbulence for a problem of type (20) have a universal nonequilibrium solution.

It may be supposed that the results obtained are valid for more general ω_k and $V_{kk'k''}$.

In conclusion the author thanks R. Z. Sagdeev for discussing the paper.

REFERENCES

1. M. Camac, A. Kantorowitz et al., "Collisionless shock waves in plasma," Nuclear Synthesis. Supplement [in Russian], Pt 2, p. 423, 1962.

2. A. A. Galeev, V. I. Karpman, and R. Z. Sagdeev, "On a soluble problem in turbulence theory," DAN SSSR, vol. 157, no. 5, p. 1088, 1964.

3. A. A. Vedenov, Questions of Plasma Theory [in Russian], Gosatomizdat, vol. 3, p. 203, 1963.

4. A. A. Galeev and V. I. Karpman, "Shock waves in a plasma situated in a magnetic field," ZhETF, vol. 44, p. 592, 1963.

5. B. B. Kadomtsev and V. I. Petviashvili, "Turbulence of a plasma situated in a magnetic field," ZhETF, vol. 43, p. 2234, 1962.

6. V. E. Zakharov, "A soluble model of weak turbulence," PMTF, no. 1, 1965.

28 October 1964 Novosibirsk